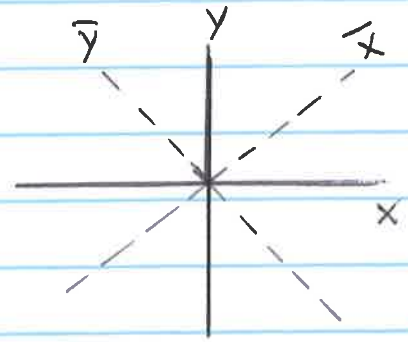
11.7 Appendix

Generic paraboloid: min, max or saddle?

$z = ax^2 + bxy + cy^2$



$x = \bar{x}\cos\theta - \bar{y}\sin\theta$

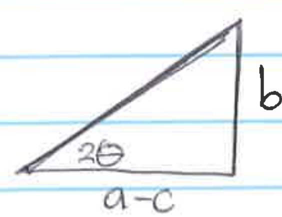
$y = \bar{x}\sin\theta + \bar{y}\cos\theta$

$$\begin{aligned} z = &\ a\left(\bar{x}^2\cos\theta - 2\bar{x}\bar{y}\sin\theta\cos\theta + \bar{y}^2\sin^2\theta\right) \\ &+ b\left(\bar{x}^2\cos\theta\sin\theta + \bar{x}\bar{y}(\cos^2\theta - \sin^2\theta) - \bar{y}^2\sin\theta\cos\theta\right) \\ &+ c\left(\bar{x}^2\sin^2\theta + 2\bar{x}\bar{y}\cos\theta\sin\theta + \bar{y}^2\cos^2\theta\right) \end{aligned}$$

$$\begin{aligned} = &\ (a\cos^2\theta + b\cos\theta\sin\theta + c\sin^2\theta)\bar{x}^2 && (A\bar{x}^2) \\ &+ (b\cos 2\theta + (c-a)\sin 2\theta)\bar{x}\bar{y} && +(B\bar{x}\bar{y}) \\ &+ (a\sin^2\theta - b\sin\theta\cos\theta + c\cos^2\theta)\bar{y}^2 && +(C\bar{y}^2) \end{aligned}$$

Now choose θ such that $\bar{x}\bar{y}$ term vanishes:

$B = b\cos 2\theta + (c-a)\sin 2\theta = 0$



$\tan 2\theta = \frac{b}{a-c} \Rightarrow \theta = \frac{1}{2}\tan^{-1}\left(\frac{b}{a-c}\right)$

At this point we have $z = A\bar{x}^2 + C\bar{y}^2$

Next work out the other trig functions for this θ

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta) = \frac{1}{2}\left(1 + \frac{a-c}{\sqrt{b^2+(a-c)^2}}\right)$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta) = \frac{1}{2}\left(1 - \frac{a-c}{\sqrt{b^2+(a-c)^2}}\right)$$

$$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta = \frac{1}{2}\frac{b}{\sqrt{b^2+(a-c)^2}}$$

Finally after a lot of algebra, we obtain

$$A = a\cos^2\theta + \frac{b}{2}\sin 2\theta + c\sin^2\theta$$

$$= \frac{a+c}{2} + \frac{(a-c)(a-c)}{2\sqrt{}} + \frac{b^2}{2\sqrt{}}$$

$$= \frac{a+c}{2} + \frac{b^2+(a-c)^2}{2\sqrt{}} = \frac{1}{2}\left(a+c+\sqrt{b^2+(a-c)^2}\right)$$

$$C = a\sin^2\theta - \frac{b}{2}\sin 2\theta + c\cos^2\theta$$

$$= \frac{a+c}{2} + \frac{(c-a)(a-c)}{2\sqrt{}} - \frac{b^2}{2\sqrt{}}$$

$$= \frac{a+c}{2} - \frac{b^2+(a-c)^2}{2\sqrt{}} = \frac{1}{2}\left(a+c-\sqrt{b^2+(a-c)^2}\right)$$

Finally, their product is

$$AC = \frac{1}{4}(a+c)^2 - (b^2 + (a-c)^2)$$

$$= \frac{1}{4}\left[a^2 + 2ac + c^2 - b^2 - a^2 + 2ac - c^2\right]$$

$$= \frac{1}{4}\left[4ac - b^2\right]$$

$$= f_{xx} f_{yy} - f_{xy} f_{yx}$$

$$= \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

$$= f_{xx} f_{yy} - (f_{xy})^2$$